UDC 539.3

INVERSE PROBLEMS OF THE PLANE THEORY OF ELASTICITY

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G. P. CHEREPANOV
(Moscow)
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In connection with the fact that failure of a structure ordinarily starts at sites of the most acute stress concentrations near cavitiies, it is of interest to determine the shape of the equally strong outlines of holes on which the technologically inevitable stress concentration would be least as compared with all other outlines.

An effective exact solution of some inverse plane problems of the theory of elasticity concerning the determination of equally strong outlines of holes is proposed. A formulation of the problem is given first and the fundamental relationships are presented. Then the general problem for any number of holes in an infinite plane is reduced to a standard Dirichlet problem for the exterior of the same number of parallel slits on a parametric plane. An effective exact solution is found by this method for the case of one and two holes as well as for the case of periodic and doubly-periodic series of holes. The question of application of the solutions obtained to the theory of a minimum weight structure is considered.

1. Formulation of the problem and fundamental relationships. Let us examine the state of stress in an infinite isotropic and homogeneous elastic plate weakened by curvilinear holes, whose number can be arbitrary. Let us assume that constant normal and tangential stress resultants

$$\sigma_n = p, \quad \tau_{nt} = \tau \tag{1.1}$$

are applied to the edges of the hole while a homogeneous field of constant stresses acts at infinity (in the case of a doubly-periodic series of holes the conditions replacing (1.2) are presented below in Sect. 5)

$$\sigma_x = \sigma_x^\infty, \quad \sigma_y = \sigma_y^\infty, \quad \tau_{xy} = \tau^\infty \tag{1.2}$$

Here x, y are rectangular Cartesian coordinates, t and n are the tangent and normal to the hole outline (generating a right-hand system nt).

Let us pose the following problem: find the shape of holes and their mutual disposition such that the tangential normal stress σ_t acting on these outlines would be a constant, identical for all the holes. Let us therefore require compliance with the following condition on all the hole outlines:

$$\sigma_t = \sigma = \text{const} \tag{1.3}$$

Such holes are called equally strong [1]. As the holes themselves, the quantity σ is to be determined. A plastic zone evidently originates simultaneously over the whole outline on the equally strong holes.

The formulation and solution of some problems of this kind exists in [1-11]. The close connection between these problems and theories of a minimum weight structure is discussed below in Sect. 6.

Let us represent the stress components [12] in terms of the Kolosov-Muskhelishvili potentials $\Phi(z)$ and $\Psi(z)$

$$\sigma_x + \sigma_y = 4\operatorname{Re}\Phi(z) \qquad (z = x + iy)$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] \tag{1.4}$$

According to (1.2), as z tends to infinity the functions $\Phi(z)$ and $\Psi(z)$ behave thus

$$\Phi(z) = {}^1/_4(\sigma_x^\infty + \sigma_y^\infty) + O(z^{-2}) \tag{1.5}$$
$$\Psi(z) = b + O(z^{-2}), \quad b = {}^1/_2(\sigma_y^\infty - \sigma_x^\infty) + i\tau^\infty$$

The case of an infinitely-connected domain is singular and should be considered separately.

Using the relationships

$$\sigma_t + \sigma_n = \sigma_x + \sigma_y \tag{1.6}$$
$$\sigma_t - \sigma_n + 2i\tau_{tn} = e^{2i\alpha}(\sigma_y - \sigma_x + 2i\tau_{xy})$$

where α is the angle between the external normal to the outline and the x-axis measured from x to n, as well as the representations (1.4), then the boundary conditions (1.1) and (1.2) on the unknown outlines L_0 of the holes can be represented as

$$4\operatorname{Re}\Phi(z) = \sigma + p, \quad z \in L_0 \tag{1.7}$$
$$\bar{z}\Phi'(z) + \Psi(z) = ae^{-2i\alpha}, \quad z \in L_0 \quad (a = {}^1/_2(\sigma - p) + i\tau) \tag{1.8}$$

If an analytic function is bounded everywhere within a domain (including at the infinitely remote point), and its real part is constant on the domain boundary, then the function itself is constant. Therefore, the solution of the boundary value problem (1.7), (1.5) for the function $\Phi(z)$ has the form

$$\Phi(z) = {}^1/_4(\sigma + p), \qquad \sigma = \sigma_x^\infty + \sigma_y^\infty - p \tag{1.9}$$

Taking account of (1.9), the boundary condition (1.8) is written as

$$e^{2i\alpha}\Psi(z) = a, \qquad z \in L_0 \tag{1.10}$$

where $\Psi(z) = b + O(z^{-2})$ as $z \to \infty$.

2. Method of solving the boundary value problem. Let us go over to the parameteric plane of the complex variable ζ by using conformal mapping performed by the analytic function $\omega(\zeta)$

$$z = \omega(\zeta) \tag{2.1}$$

Let us recall the fundamental facts [13] from the theory of conformal mapping of multiconnected domains: (a) every n-connected domain, including the infinitely remote point, can always be mapped conformally on the exterior of some n slits parallel to the real axis with the infinitely remote points coincident, (b) for $n \geqslant 3$ this mapping is unique if the behavior of the mapping function at infinity $\omega(\zeta) = \zeta + O(1)$ as $\zeta \to \infty$ is specified.

Let us consider the desired function $\omega(\zeta)$ to give a mutually one-to-one correspondence between the elastic domain on the z plane and the exterior of the corresponding number of slits M on the ζ plane, which are parallel to the real axis (see Fig. 1, where the case of a doubly-connected domain is shown).

We determine $e^{2i\alpha}$. We give an increment to the point z in the normal direction to the outline L_0

$$dz = e^{i\alpha} |dz| \tag{2.2}$$

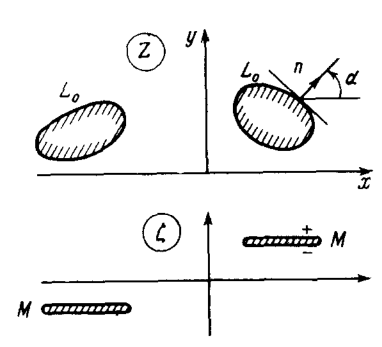

Fig. 1

By virtue of the conformality of the mapping, the corresponding point on the ζ plane will move along the normal to the slit, i.e. to the real axis

$$d\zeta = \pm i |d\zeta| \tag{2.3}$$

Using (2.2) and (2.3) we find

$$e^{i\alpha} = \frac{dz}{|dz|} = \frac{\omega'(\zeta)\, d\zeta}{|\omega'(\zeta)|\, |d\zeta|} = \pm i \frac{\omega'(\zeta)}{|\omega'(\zeta)|}$$

Therefore

$$e^{2i\alpha} = -\omega'(\zeta) / \overline{\omega'(\zeta)} \tag{2.4}$$

By using (2.4) the boundary value problem (1.11) is now written thus:

$$-\psi(\zeta)\,\omega'(\zeta) = a\overline{\omega'(\zeta)}, \qquad \zeta \in M \tag{2.5}$$

Here

$$\psi(\zeta) = \Psi[\omega(\zeta)]$$

The functions $\psi(\zeta)$ and $\omega(\zeta)$ are to be determined from the boundary value problem (2.5). Let us take the real and imaginary parts in the expression (2.5), whereupon we obtain

$$\operatorname{Re} F'(\zeta) = 0, \qquad \zeta \in M \tag{2.6}$$

$$\operatorname{Im} G'(\zeta) = 0, \qquad \zeta \in M \tag{2.7}$$

Here

$$F'(\zeta) = \psi(\zeta)\omega'(\zeta) + \bar{a}\omega'(\zeta), \quad G'(\zeta) = \psi(\zeta)\omega'(\zeta) - \bar{a}\omega'(\zeta) \tag{2.8}$$

The functions $F'(\zeta)$ and $G'(\zeta)$ are analytic everywhere in the exterior of the slits M. They are bounded in the neighborhood of the infinitely remote point since the functions $\psi(\zeta)$ and $\omega'(\zeta)$ are bounded as $\zeta \to \infty$.

The behavior of the functions $F'(\zeta)$ and $G'(\zeta)$ at the ends of the slits M is determined by the requirements imposed on the desired hole outlines L_0. Let us require that all the contours L_0 be smooth, i.e. should not contain cusps and corners. Under this additional condition the function $\psi(\zeta)$ is bounded everywhere in the neighborhood of points of the slits M, but the function $\omega'(\zeta)$ is bounded everywhere with the exception of the ends of the slits M in whose neighborhoods $\omega'(\zeta)$ evidently have a power singularity of order $^1/_2$. In conformity with this condition, on the basis of (2.8), the analytic functions $F'(\zeta)$ and $G'(\zeta)$ are bounded in the whole ζ plane with the exception of the ends of the slits M at which they have a power-law singularity of order $^1/_2$. For example, if the point ζ_M is an endpoint of one of the slits M. then

$$F'(\zeta) = O\left(\frac{1}{\sqrt{\zeta - \zeta_M}}\right), \quad G'(\zeta) = O\left(\frac{1}{\sqrt{\zeta - \zeta_M}}\right) \quad \text{for } \zeta \to \zeta_M \tag{2.9}$$

The boundary value problems (2.6) and (2.7) are classical Dirichlet problems for the exterior of slits, where the solution of the problems is sought in the class of functions bounded at infinity and having a singularity of the form (2.9) at the ends of the slits. Namely, the hydrodynamic problem of the flow over the cascade of profiles by a potential ideal incompressible weightless fluid stream results in such a mathematical problem

[14]. A complex velocity potential of the fluid stream hence corresponds to the functions F and G.

Therefore, the boundary value problem (2.6) or (2.7) is completely analogous, in mathematical respects, to the following hydrodynamic problem: a potential, circulation-free, ideal incompressible fluid stream flows over a system of M zero-thickness flat profiles in the ζ plane (the stream velocity is bounded at infinity); find the complex flow potential.

After the functions $F(\zeta)$ and $G(\zeta)$ have been found, the desired functions $\psi(\zeta)$ and $\omega(\zeta)$ are determined by using (2.8) according to the following formulas:

$$\omega(\zeta) = \frac{1}{2\bar{a}} \int [F'(\zeta) - G'(\zeta)]\, d\zeta = \frac{1}{2\bar{a}} [F(\zeta) - G(\zeta)] + C_0 \qquad (2.10)$$

$$\psi(\zeta) = \bar{a}\,\frac{F'(\zeta) + G'(\zeta)}{F'(\zeta) - G'(\zeta)} \qquad (2.11)$$

where (C_0 is an arbitrary constant.

Let us clarify yet another condition which results from the physical requirement of no dislocations upon traversing the outline of each hole. The following condition which the function $\omega(\zeta)$ should satisfy

$$\oint \omega'(\zeta)\, \psi(\zeta)\, d\zeta = 0 \qquad (2.12)$$

is hence obtained easily from the expression for the complex displacement vector [12] and (1.9). The contour integral is here taken around the contour enclosing one of the slits M. The number of conditions (2.12) equals the number of slits. Deforming the contour of integration at the upper and lower edges of the appropriate slit, and using (2.5), we hence obtain

$$\oint \overline{\omega'(\zeta)}\, d\zeta = 0 \qquad (2.13)$$

Since the slits M are parallel to the imaginary axis, then $d\zeta = \overline{d\zeta}$, therefore, condition (2.13) means that the function $\omega(\zeta)$ should be unique upon traversing any of the slits M. However, this condition is already contained in the conformal mapping requirements realized by the function $\omega(\zeta)$. Hence, the requirement of no-dislocations in a multiconnected elastic domain results in this case from the required condition of uniqueness of the function $\omega(\zeta)$.

According to (1.4), (1.9), (1.10), (2.5) and (2.8), the field of elastic stresses is found from the following relationships:

$$\sigma_x + \sigma_y = \sigma_x^\infty + \sigma_y^\infty, \qquad \sigma_y - \sigma_x + 2i\tau_{xy} = \frac{F'(\zeta) + G'(\zeta)}{\omega'(\zeta)} \qquad (2.14)$$

Hence, the approach developed permits finding the effective solution of inverse problems of the plane theory of elasticity for any number and for any arrangement of holes.

3. One or two holes. A doubly- and triply-connected domain can always be mapped conformally on the exterior of a correponding number of slits along the real axis [15].

One hole. In the case of one hole, there will be one slit on the plane which can be considered the slit $(-1, +1)$ along the real axis without loss of generality.

For $\zeta \to \infty$ we have $\omega(\zeta) = c_1\zeta + O(\zeta^{-1})$. The quantity c_1 can be considered real. According to the Riemann theorem, this condition together with assignment of the slit length exhausts the possible arbitrariness in the description of the conformal transformation of the two given domains.

According to (2.8), we have

for $\zeta \to \infty$

$$F'(\zeta) = (b+\bar{a})c_1 + O(\zeta^{-2}), \quad G'(\zeta) = (b-\bar{a})c_1 + O(\zeta^{-2}) \quad (3.1)$$

The solution of the boundary value problems (2.6) and (2.7) for the exterior of the slit mentioned is the following under conditions (2.9) and (3.1) [14]:

$$F'(\zeta) = ic_1 \operatorname{Im}(b+\bar{a}) + \frac{c_1\zeta \operatorname{Re}(b+\bar{a}) + d_1}{\sqrt{\zeta^2-1}} \quad (3.2)$$

$$G'(\zeta) = c_1 \operatorname{Re}(b-\bar{a}) + \frac{ic_1\zeta \operatorname{Im}(b-\bar{a}) + id_2}{\sqrt{\zeta^2-1}}$$

Here we have $\sqrt{\zeta^2-1} = \zeta + O(\zeta^{-1})$ as $\zeta \to \infty$. The real constants d_1 and d_2 are arbitrary. Integrating (3.2) we find

$$F(\zeta) = ic_1\zeta \operatorname{Im}(b+\bar{a}) + c_1\sqrt{\zeta^2-1}\operatorname{Re}(b+\bar{a}) + d_1 \ln(\zeta+\sqrt{\zeta^2-1}) \quad (3.3)$$

$$G(\zeta) = c_1\zeta \operatorname{Re}(b-\bar{a}) + ic_1\sqrt{\zeta^2-1}\operatorname{Im}(b-\bar{a}) + id_2 \ln(\zeta+\sqrt{\zeta^2-1})$$

Hence, setting $C_0 = 0$ we obtain by using (2.10)

$$\omega(\zeta) = \frac{c_1}{2}(m_1\zeta + m_2\sqrt{\zeta^2-1}) + \frac{d_1 - id_2}{2\bar{a}} \ln(\zeta + \sqrt{\zeta^2-1}) \quad (3.4)$$

$$(m_1 = 1 - \bar{b}/\bar{a}, \quad m_2 = 1 + \bar{b}/\bar{a})$$

The function $\omega(\zeta)$ should be unique in the exterior of the slit $(-1, +1)$. According to the solution (3.4), this condition is satisfied only if $d_1 = d_2 = 0$, which is henceforth assumed.

The equation of the hole outline in parameteric form is obtained from (3.4) for $d_1 = d_2 = 0$

$$x = \tfrac{1}{2} c_1 (\xi \operatorname{Re} m_1 \mp \sqrt{1-\xi^2} \operatorname{Im} m_2) \quad (3.5)$$

$$y = \tfrac{1}{2} c_1 (\xi \operatorname{Im} m_1 \pm \sqrt{1-\xi^2} \operatorname{Re} m_2) \quad (-1 < \xi < 1)$$

or in complex form

$$z = \tfrac{1}{2} c_1 (m_1\xi \pm im_2\sqrt{1-\xi^2}) \quad (3.6)$$

where ξ is a real parameter less than unity (the upper sign in (3.5) corresponds to the upper edge of the slit, and the lower sign to the lower edge). Eliminating the parameter ξ in (3.5), we arrive at a second order curve, an ellipse, since on the basis of (3.6) finite values of the complex vector of the curve correspond to any values of the parameter ξ less than unity in absolute value.

Let us find the fundamental parameters of this ellipse, which is the desired hole outline. The foci of the ellipse are branch points on the two-sheeted Riemann surface of the function $\zeta = \zeta(z)$, the inverse of the function (3.4). We hence find the complex vectors of the foci

$$z_F = \pm \tfrac{1}{2} c_1 \sqrt{m_1^2 - m_2^2} \quad (3.7)$$

The center of the ellipse (the middle of the segment connecting the foci) coincides with the origin. The angle the major axis of the ellipse encloses with the abscissa axis is

$$\alpha_F = \tfrac{1}{2} \arg(m_1^2 - m_2^2) \quad (3.8)$$

According to (3.6), the point $z_1 = \tfrac{1}{2} c_1 m_1$, lies on the ellipse. According to the focal property of the ellipse, the sum of the distances from this point to the foci equals the major diameter of the ellipse $2a_1$, i.e.

$$a_1 = {}^1/_4 c_1 (| \sqrt{m_1^2 - m_2^2} + m_1 | + | m_1 - \sqrt{m_1^2 - m_2^2} |) \tag{3.9}$$

The minor semi-axis of the ellipse equals $\sqrt{a_1^2 - |z_F|^2}$.

Therefore, the outlines of the desired hole are a family of similar ellipses (since c_1 is an arbitrary real parameter), whose orientation and fundamental parameters are given by (3.7)–(3.9). In the case $\tau^\infty = \tau = 0$, when the quantities m_1 and m_2 are real, the result corresponds to that obtained earlier in [1] by another method.

Two holes. Let us map the exterior of two holes conformally onto the exterior of two slits (λ_1, λ_2) and (λ_3, λ_4) along the real axis of the ζ plane with the infinitely remote points coincident. Without limiting the generality, the coefficient c_1 can be considered real and positive in the conditions (3.1) at infinity. The solution of the Dirichlet problem (2.6) and (2.7) for the slits mentioned is, under the conditions (3.1) at infinity,

$$F'(\zeta) = ic_1 \operatorname{Im}(b + \bar{a}) + \frac{1}{R}[c_1\zeta^2 \operatorname{Re}(\bar{a} + b) + d_1\zeta + d_2] \tag{3.10}$$

$$G'(\zeta) = c_1 \operatorname{Re}(b - \bar{a}) + \frac{1}{R}[ic_1\zeta^2 \operatorname{Im}(b - \bar{a}) + id_3\zeta + id_4]$$

$$R = \sqrt{(\zeta - \lambda_1)(\zeta - \lambda_2)(\zeta - \lambda_3)(\zeta - \lambda_4)}$$

Here $d_1 - d_4$ are real arbitrary constants, and the square root in (3.10) behaves as $\zeta^2 + O(\zeta)$ for $\zeta \to \infty$. In this case the constants can be set $\lambda_1 = 0$ and $\lambda_2 = 1$. In all, seven undetermined real constants are contained in (3.10). Four of them are determined from the two complex uniqueness conditions for the functions $\omega(\zeta)$ (see (2.14)). The constant c_1, which gives the scale in the physical z plane, is undetermined by the formulation of the problem. Two constants remain which are determined by imposing added requirements on the desired outlines and their mutual disposition. Therefore, in the general nonsymmetric case the outlines of the desired holes form a family dependent on two arbitrary real parameters. We omit the awkward result of integrating in terms of elliptic functions.

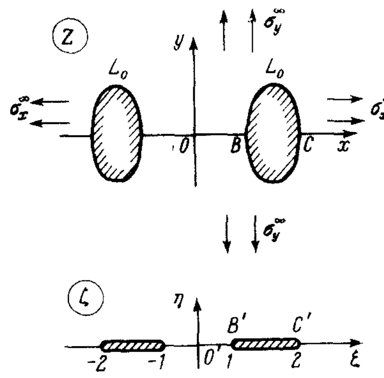

Fig. 2

Let us consider the symmetric case when $\tau^\infty = \tau = 0$ (Fig. 2) in more detail. In this case the outlines of both holes are symmetric relative to the abscissa and ordinate axes. For definiteness, let us consider one hole to be located entirely in the left half-plane $(x < 0)$, and the other to be symmetrically in the right half-plane $(x > 0)$. In the symmetric case under consideration, we can (without limiting the generality) set in (3.10)

$$\lambda_1 = -2, \ \lambda_2 = -1, \ \lambda_3 = 1, \ \lambda_4 = 2, \tag{3.11}$$

$$d_1 = d_3 = d_4 = 0, \ (\operatorname{Im} a = 0, \ \operatorname{Im} b = 0)$$

This last relationship is a result of the symmetry of the problem and the condition $dy / d\xi = 0$ for $|\xi| < 1$. We consequently obtain

$$F'(\zeta) = \frac{c_1\zeta^2(b + a) + d_2}{\sqrt{(\zeta^2 - 1)(\zeta^2 - 4)}}, \qquad G'(\zeta) = c_1(b - a) \tag{3.12}$$

We hence evaluate the function $\omega(\zeta)$ by using (2.10) (the constant C_0 is selected from the condition that $\omega(0) = 0$)

$$\omega(\zeta) = \frac{c_1}{2a}\left[\zeta(a-b) + 2(a+b)E\left(\arcsin\zeta, \frac{1}{2}\right) - (a+b)\left(2+\frac{1}{2}d_2\right)F\left(\arcsin\zeta, \frac{1}{2}\right)\right] \tag{3.13}$$

Here F and E are the elliptic integrals of the first and second kinds, respectively. The coordinates of the point B on the outline of the right-hand hole (Fig. 2) can be obtained by means of (3.13) for $\zeta = 1$

$$x_B = \frac{c_1}{2a}\left[a - b + 2(a+b)E\left(\frac{1}{2}\right) - K\left(\frac{1}{2}\right)(a+b)\left(2+\frac{1}{2}d_2\right)\right] \tag{3.14}$$

It is convenient to determine the shape of the outline directly by means of (3.12) and (2.10); we find [16] (for $x > 0$ and $y > 0$)

$$x = x_B + \frac{1}{2}c_1\left(1-\frac{b}{a}\right)(\xi - 1) \tag{3.15}$$

$$y = -\frac{1}{2}c_1\left(1+\frac{b}{a}\right)\int_1^{\xi}\frac{(\xi^2+d_2)\,d\xi}{\sqrt{(\xi^2-1)(4-\xi^2)}} =$$

$$-\frac{1}{2}c_1\left(1+\frac{b}{a}\right)\left[2E\left(\varphi, \frac{\sqrt{3}}{2}\right) + \frac{1}{2}d_2F\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{1}{\xi}\sqrt{(4-\xi^2)(\xi^2-1)}\right]$$

$$\left(\varphi = \arcsin\frac{2\sqrt{\xi^2-1}}{\xi\sqrt{3}}, \quad 1 < \xi < 2\right)$$

For $\xi = 2$, i.e. for $\varphi = \pi/2$, the y coordinate of the hole outline (at the point C, see Fig. 2) should equal zero. This condition serves to determine the unknown constant d_2; by using the second formula in (3.15) we find

$$d_2 = -\frac{4E(\sqrt{3}/2)}{K(\sqrt{3}/2)} = -2.246 \tag{3.16}$$

Inserting this value of d_2 into (3.14) and (3.15), we finally obtain

$$x_B = -c_1(0.23 + 1.23\,b/a), \quad x = x_B + {}^1\!/_2\,c_1(1 - b/a)(\xi - 1) \tag{3.17}$$
$$y = -{}^1\!/_2\,c_1(1 + b/a)[2E(\varphi, \sqrt{3}/2) - 1.123F(\varphi, \sqrt{3}/2) - \xi^{-1}\sqrt{(4-\xi^2)(\xi^2-1)}]$$

Therefore, in the case of a symmetric arrangement of two equally strong holes, the solution is determined to the accuracy of one arbitrary positive constant c_1 which yields the scale. The hole outline is defined by (3.17).

Let us note certain constraints which the external loads should satisfy so that the solution (3.17) would have physical meaning. According to (3.17) itself, it is necessary for this that the inequalities

$$1 > b/a, \quad 1 > -b/a, \quad b/a < -0.23/1.23 \approx -0.187$$
$$-1 < b/a < -0.187$$

must hold, where

$$b / a = (\sigma_y^\infty - \sigma_x^\infty) / (\sigma_x^\infty + \sigma_y^\infty - 2p) \tag{3.18}$$

As is easy to see, this inequality results in the following conditions (a) or (b) for the existence of the desired solution:

$$\text{(a)} \quad \sigma_x^\infty > \sigma_y^\infty > p, \qquad \sigma_x^\infty - \sigma_y^\infty > 0.315\,(\sigma_x^\infty - p)$$
$$\text{(b)} \quad \sigma_x^\infty < \sigma_y^\infty < p, \qquad \sigma_x^\infty - \sigma_y^\infty < 0.315\,(\sigma_x^\infty - p)$$

Outlines of a family of equally strong holes are constructed by means of (3.15) in Fig. 3 for different values of b / a (for $x > 0$ and $y > 0$).

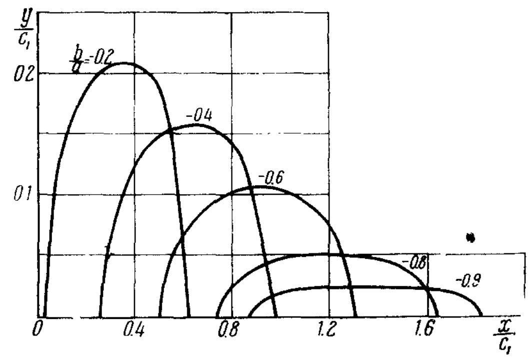

Fig. 3

The approximate solution of the problem considered by the small parameter method was attempted in [7].

4. Periodic problem. If all the hole outlines have a common axis of symmetry which intersects them, then the domain of any connectedness located outside these holes is mapped conformally on the exterior of slits along the same line (*). In this case the Dirichlet problem has a simple closed solution.

Let us consider the periodic problem when identical holes are arranged with some period $2L$ along the x-axis, which is not certainly their axis of symmetry because of the presence of shear stresses (Fig. 4).

Let us go over to the exterior of a periodic system of M slits along the real axis of the ζ plane by using the conformal mapping $z = \omega(\zeta)$ (the lattice period can be taken equal to π, see Fig. 4). As before, we use condition (3.1) as the condition at infinity; it should be kept in mind that the second terms in these formulas describe an exponential decrease at infinity in this case, and not a power-law decrease as in the case of a finite number of holes. In the case under consideration, the constant c_1 can evidently be considered real and positive. The solution of the boundary value problems (2.6) and (2.7) under the additional conditions (2.9) and (3.1) is found easily on the infinite-sheeted Riemann surface of the function $\sin\zeta$ slit along the segment $(\sin e_1, \sin e_2)$ of the real axis on both sheets. The function $\sin\zeta$ conformally maps the exterior of the periodic system of M slits of the ζ plane onto the mentioned Riemann surface.

*) The same is valid for multiconnected domains possessing cyclic symmetry.

The general solution of these problems is

$$F'(\zeta) = ic_1 \operatorname{Im}(b + \bar a) + \frac{1}{R}[c_1 \sin\zeta \operatorname{Re}(b + \bar a) + d_1] \tag{4.1}$$
$$G'(\zeta) = c_1 \operatorname{Re}(b - \bar a) + \frac{1}{R}[ic_1 \sin \operatorname{Im}(b - \bar a) + id_2]$$
$$R = \sqrt{(\sin\zeta - \sin e_1)(\sin\zeta - \sin e_2)}$$

Here c_1, d_1, d_2, e_1, e_2 are undetermined real constants; as $\sin\zeta \to \infty$ the root R in (4.1) behaves as $\sin\zeta + O(\sin^{-1}\zeta)$. The function $\omega(\zeta)$ is determined by (2.10) under the additional condition $\omega(-\pi/2) = -L$ to find the constant C_0 (see Fig. 4):

$$\omega(\zeta) = -L + \frac{1}{2\bar a}\int_{-\pi/2}^{\zeta} [F'(\zeta) - G'(\zeta)]\, d\zeta \tag{4.2}$$

The function $\omega(\zeta)$ must satisfy two additional conditions: (a) the condition of uniqueness of $\omega(\zeta)$ upon traversing the slit (e_1, e_2); (b) the condition of correspondence of the points C and C', having the form $\omega(\pi/2) = L$ (see Fig. 4). These conditions are to seek four (out of the five) undetermined coefficients in (4.1) and (4.2). Therefore, in the general nonsymmetric case of the periodic problem, the solution is determined to the accuracy of one arbitrary constant, i.e. equally strong outlines possess a family dependent on one free parameter.

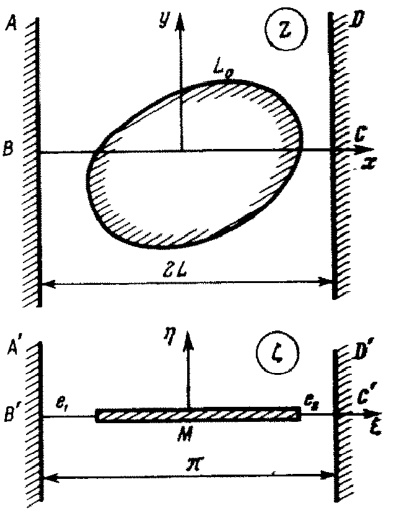

Fig. 4

Let us consider the symmetric case when $\tau^\infty = \tau = 0$ in more detail. In this case the outline of the desired hole L_0 in the fundamental period (Fig. 4) is symmetric relative to the abscissa and ordinate axes, and its corresponding segment M on the ζ plane is a symmetric slit along $(-e_0, e_0)$ of length $2e_0$. By virtue of the symmetry mentioned, we can set in (4.1)

$$e_1 = -e_0, \quad e_2 = e_0, \quad d_1 = d_2 = 0$$
$$(\operatorname{Im} a = \operatorname{Im} b = 0)$$

We therefore find

$$F'(\zeta) = \frac{c_1(a+b)\sin\zeta}{\sqrt{\sin^2\zeta - \sin^2 e_0}}, \quad G'(\zeta) = c_1(b-a) \tag{4.3}$$

$$\omega(\zeta) = -L + {}^1/_2\, c_1(1 - b/a)(\zeta + \pi/2) - {}^1/_2\, c_1(1 + b/a) \arcsin(\cos\zeta / \cos e_0) \tag{4.4}$$
$$(\omega(-\pi/2) = -L)$$

From the condition of correspondence of the points C and C' in Fig. 4, i.e. $\omega(\pi/2) = L$, we obtain by using (4.4)

$$c_1 = -(2aL)/(\pi b) \tag{4.5}$$

Let us find the diameter $2x_0$ of an equally strong hole in the $y = 0$ section; setting $\zeta = e_0$ in (4.4), we obtain

$$x_0 = \frac{1}{\pi} e_0 L \left(1 - \frac{a}{b}\right) \tag{4.6}$$

By using (4.4)–(4.6) we write the equation of an equally strong outline in the following parametric form (for $y > 0$, $|x| < L$):

$$x = \frac{1}{\pi} L \left(1 - \frac{a}{b}\right) \xi \tag{4.7}$$

$$y = -\frac{1}{\pi} L \left(1 + \frac{a}{b}\right) \ln\left(\frac{\cos\xi}{\cos e_0} + \sqrt{\frac{\cos^2\xi}{\cos^2 e_0} - 1}\right)$$

$$(-e_0 < \xi < e_0)$$

The family of curves (4.7) depends on one positive parameter e_0, smaller than $\pi / 2$. We find the diameter $2y_0$ of an equally strong hole in the section $x = 0$; setting $\xi = 0$ in the second formula in (4.7), we obtain

$$y_0 = -\frac{1}{\pi} L \left(1 + \frac{a}{b}\right) \ln \frac{1 + \sin e_0}{\cos e_0} \tag{4.8}$$

In conformity with the physical meaning of the problem, the quantities x_0 and y_0 must be positive (moreover, evidently $x_0 < L$); hence on the basis of (4.6) and (4.8) the external loads must satisfy the following condition

$$-1 < \frac{b}{a} < -\frac{e_0}{\pi - e_0} \tag{4.9}$$

where b / a is given by (3.18), As is easy to see, this inequality results in the following conditions (a) or (b) for the existence of the desired solution:

$$\text{(a)} \quad p > \sigma_y^\infty, \quad (\pi - 2e_0)\sigma_x^\infty < \pi\sigma_y^\infty - 2pe_0 \tag{4.10}$$

$$\text{(b)} \quad \sigma_y^\infty > p, \quad (\pi - 2e_0)\sigma_x^\infty > \pi\sigma_y^\infty - 2pe_0$$

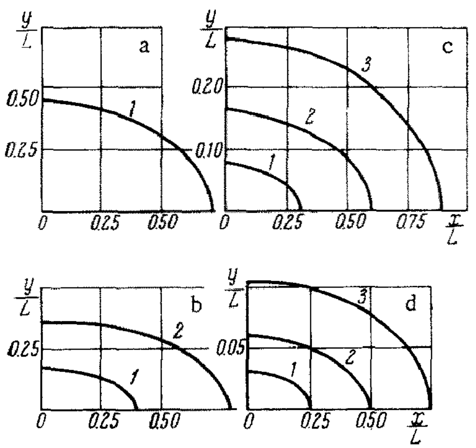

Fig. 5

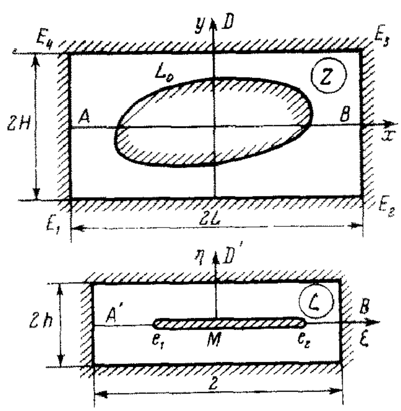

Fig. 6

Let us examine the limit case of one hole in an infinite plane. This case is obtained from the periodic solution found by the following passage to the limit in (4.3)–(4.8):

$$L \to \infty, \quad e_0 \to 0, \quad \zeta \to 0, \quad \zeta L = -\frac{\pi b}{2u} C_1 \zeta_1, \quad e_0 L = -\frac{\pi b}{2a} C_1 \tag{4.11}$$

where C_1 (specified arbitrarily) is a positive constant, and ζ_1 is a new complex variable. In particular, the hole outline becomes an ellipse

$$x = {}^1/_2 C_1 (1 - b / a) t \qquad \qquad (4.12)$$
$$y = {}^1/_2 C_1 (1 + b / a) \sqrt{1 - t^2} \quad (-1 < t < 1)$$

The latter solution exists under the condition $| b / a | < 1$ (cf. corresponding condition (4.9) for the periodic problem).

Outlines of a family of equally strong holes are constructed by means of (4.7) in Fig. 5 for different values of e_0 ($e_0 = 20°, 40°, 60°$ correspond to curves 1–3) and $b/a = -0.2$ (a), -0.4 (b), -0.6 (c), -0.8 (d) (for $L > x > 0,\ y > 0$).

An approximate solution of the problem considered by the small parameter method was attempted in [6, 11].

5. Doubly-periodic problem. Let a system of equally-strong holes form a doubly-periodic rectangular lattice with period $2(L + iH)$ in the z plane (see Fig. 6, where the fundamental period is shown). For reasons of convenience in the solution, let us temporarily consider stresses to be given at the point $x = 0,\ y = H$, (and the other corresponding points), i.e. for $z \to iH$

$$\sigma_x = \sigma_x^\infty, \quad \sigma_y = \sigma_y^\infty, \quad \tau_{xy} = \tau^\infty \qquad (5.1)$$

Here σ_x^∞, σ_y^∞, τ^∞ are certain quantities.

Let the conformal mapping $z = \omega(\zeta)$ transform the exterior of a doubly-periodic system of slits in the ζ plane in the elastic domain of the z plane, with correspondence of three points of the fundamental periods (see Fig. 6)

$$A \leftrightarrow A', \quad B \leftrightarrow B', \quad D \leftrightarrow D' \qquad (5.2)$$

The period of the rectangular lattice in the ζ plane can be considered equal to $2 + 2hi$ without limiting the generality.

The desired function $\omega(\zeta)$ is defined uniquely by the requirements mentioned (for a given hole outline L_0).

The exterior of the outline L_0 in the fundamental period of the z plane is mapped onto the exterior of the slit (e_1, e_2) along the real axis of the ζ plane (also in the fundamental period, see Fig. 6).

According to (5.1), (5.2), (1.4) and (2.8), we have $\omega(\zeta) = iH + c_1(\zeta - ih) + \ldots$ as $\zeta \to ih$.

$$\psi(\zeta) = b + O(\zeta - ih), \quad F'(\zeta) = (b + \bar{a})c_1 + O(\zeta - ih) \qquad (5.3)$$
$$G'(\zeta) = (b - \bar{a})c_1 + O(\zeta - ih)$$

Here c_1 is some positive quantity and the notation a and b agrees with that used earlier.

Let the modulus k be defined by the equation

$$K(\sqrt{1 - k^2}) = hK(k) \qquad (5.4)$$

and the quantity

$$\alpha = 1 / K(k) \qquad (5.5)$$

The elliptic sine $w = sn(\zeta / \alpha, k)$ maps the exterior of the mentioned doubly-periodic system of slits in the ζ plane onto the infinitely-sheeted Riemann surface w slit along the segment (w_1, w_2) of the real axis on all the sheets, where

$$w_1 = sn(e_1 / \alpha, k), \quad w_2 = sn(e_2 / \alpha, k) \qquad (5.6)$$

where the point $\zeta = ih$ corresponds to the infinitely remote point on the principal sheet of the Riemann surface. Using the Riemann surface w and the condition (5.3), the general solution of the boundary value problem (2.6) and (2.7) can be written thus:

$$F'(\zeta) = ic_1 \operatorname{Im}(b + \bar{a}) + \frac{1}{R}[c_1 w \operatorname{Re}(b + \bar{a}) + d_1] \qquad (5.7)$$

$$G'(\zeta) = c_1 \operatorname{Re}(b - \bar{a}) + \frac{1}{R}[ic_1 w \operatorname{Im}(b - \bar{a}) + id_2]$$

$$R = \sqrt{(w - w_1)(w - w_2)}, \qquad w = \operatorname{sn}(\zeta / \alpha, k)$$

Here c_1, d_1, d_2, w_1, w_2 are undetermined real constants. As $w \to \infty$ the root R in (5.8) behaves as $w + O(1)$. The function $\omega(\zeta)$ is defined by means of (2.10), where the constant C_0 is found from the condition that the points A and A' correspond. Four of the above-mentioned constants are sought from the conditions that the function $\omega(\zeta)$ is unique as the slit (e_1, e_2) is traversed and from the condition that the points B and B' correspond. The constant h is found from the condition that $\omega(\zeta) \to iH$ as $\zeta \to ih$. Therefore, in the general nonsymmetric case of the periodic problem, the solution is determined to the accuracy of one arbitrary constant, i.e. the lattice of equally strong outlines forms a family dependent on one free parameter.

Up to now the state of stress has been considered given at the point $z = iH$, which does not conform to the physical substance of the problem. We determine the additional conditions which are adequate to the formulation of the problem and can determine the constants σ_x^∞, σ_y^∞ and τ^∞ in the solution obtained.

Let us consider an imaginary elastic rectangle with the sides $2mL$ and $2nH$ consisting of mn perforated rectangles $E_1E_2E_3E_4$ (m and n are integers which we shall consider quite large). Constant normal stresses N_x, N_y and constant tangential loads N_{xy}, N_{yx} are applied to the sides of this rectangle. The edge effect damps out at distances equal approximately to two lattice periods, and the state of stress becomes practically identical to that which holds in an ideal infinite lattice.

The equilibrium conditions of all the external loads acting on the rectangle mentioned. result in the following conditions for $m \to \infty$ and $n \to \infty$:

$$\int_{E_3E_4} \sigma_y \, dx = \int_{E_1E_2} \sigma_y \, dx = 2LN_y \qquad (5.8)$$

$$\int_{E_1E_4} \sigma_y \, dy = \int_{E_2E_3} \sigma_x \, dy = 2HN_x$$

$$\int_{E_3E_4} \tau_{xy} \, dx = \int_{E_1E_2} \tau_{xy} \, dx = 2LN_{xy}$$

$$\int_{E_1E_4} \tau_{xy} \, dy = \int_{E_2E_3} \tau_{xy} \, dy = 2HN_{yx}$$

Moreover, the equilibrium equation in moments

$$4HL(N_{xy} - N_{yx}) = M, \qquad M = \tau \oint_{L_0} |\mathbf{r} \times d\mathbf{s}| \qquad (5.9)$$

should be satisfied. Here M is the total moment caused by the external tangential loads τ on the hole outline L_0 ($\mathbf{r}$ is the radius vector of the curve L_0 and $d\mathbf{s}$ is its arclength vector).

Let us consider the symmetric case, when $\tau = 0$, $N_{xy} = N_{yx} = 0$, in more detail. In this case, the outline of the desired hole L_0 in the fundamental period (see Fig. 6) is symmetric relative to the x and y axes, and its corresponding outline M on the ζ plane is a slit along $(-e_0, e_0)$. Symmetry relative to the vertical and horizontal axes is conserved on the ζ plane and on the surface w. In the case under consideration, we can set in (5.7)

$$e_1 = -e_0, \quad e_2 = e_0, \quad d_1 = d_2 = 0, \quad \tau^\infty = 0$$
$$(\operatorname{Im} a = \operatorname{Im} b = 0)$$

We therefore obtain

$$F'(\zeta) = \frac{c_1(a+b)\operatorname{sn}(\zeta/\alpha, k)}{\sqrt{\operatorname{sn}^2(\zeta/\alpha, k) - \operatorname{sn}^2(e_0/\alpha, k)}}, \quad G'(\zeta) = c_1(b-a) \qquad (5.10)$$

$$\omega(\zeta) = -L + \frac{1}{2a}\int_{-1}^{\zeta} [F'(\zeta) - G'(\zeta)]\, d\zeta = -L + \qquad (5.11)$$
$$\frac{1}{2}c_1\left(1 - \frac{b}{a}\right)(\zeta + 1) + \frac{1}{2}c_1\left(1 + \frac{b}{a}\right) I(\zeta)$$

Here

$$I(\zeta) = \int_{-1}^{\zeta} \frac{\operatorname{sn}(\zeta/\alpha, k)\, d\zeta}{\sqrt{\operatorname{sn}^2(\zeta/\alpha, k) - \operatorname{sn}^2(e_0/\alpha, k)}} = \qquad (5.12)$$
$$\frac{\alpha}{\operatorname{dn}(e_0/\alpha, k)} \int_{\delta}^{t} \frac{dt}{\sqrt{(t^2 - \delta^2)(1 - t^2)}}$$
$$t = \frac{\operatorname{dn}(\zeta/\alpha, k)}{\operatorname{dn}(e_0/\alpha, k)}, \qquad \delta = \frac{\sqrt{1 - k^2}}{\operatorname{dn}(e_0/\alpha, k} \quad (\delta < 1)$$

The root $\sqrt{(t^2 - \delta^2)(1 - t^2)}$ in the complex t plane is a function which is analytic in the exterior of the slits along the real axis $(-1, -\delta)$ and $(\delta, 1)$; that branch of the function is taken which is positive on the upper edge of the slit $(\delta, 1)$ on the real axis. The relationships from the theory of elliptic functions were used (for brevity, we omit the second argument)

$$\operatorname{dn}^2 \zeta = 1 - k^2 \operatorname{sn}^2 \zeta, \qquad \operatorname{cn}^2 \zeta = 1 - \operatorname{sn}^2 \zeta \qquad (5.13)$$
$$d \operatorname{dn}\zeta / d\zeta = -k^2 \operatorname{sn} \zeta \operatorname{cn} \zeta$$

By using the elliptic integral of the first kind, the integral in (5.12) can be written thus:

$$I(\zeta) = \frac{\alpha}{\operatorname{dn}(e_0/\alpha, k)} F\left(\arcsin \frac{\sqrt{t^2 - \delta^2}}{t\sqrt{1 - \delta^2}}, \sqrt{1 - \delta^2}\right) \qquad (5.14)$$

On the basis of (5.11) and (5.14), the hole diameter $2x_0$ in the $y = 0$ section of the z plane is easily found from the condition $\omega(-e_0) = -x_0$; we obtain

$$x_0 = L - \frac{1}{2}c_1\left(1 - \frac{b}{a}\right)(1 - e_0) + \frac{1}{2}c_1\left(1 + \frac{b}{a}\right)\frac{\alpha K(\sqrt{1 - \delta^2})}{\operatorname{dn}(e_0/\alpha, k)} \qquad (5.15)$$

From the correspondence between the points B and B' in Fig. 6 (i.e. $\omega(1) = L$), we find by using (5.11)–(5.14)

$$2L = c_1\left(1 - \frac{b}{a}\right) - 2c_1\left(1 + \frac{b}{a}\right)\frac{\alpha K(\sqrt{1 - \delta^2}}{\operatorname{dn}(e_0/\alpha, k)} \qquad (5.16)$$

From the correspondence between the points D and D' in Fig. 6 (i.e. $\omega(ih) = iH$), we find by using (5.11)–(5.14)

for $\zeta \to ih,\ t \to \infty$

$$I = \frac{\alpha}{\mathrm{dn}\,(e_0/\alpha,\,k)}\left[\int_\delta^1 \frac{dt}{\sqrt{(t^2-\delta^2)(1-t^2)}} + \int_1^\infty \frac{dt}{\sqrt{(t^2-\delta^2)(1-t^2)}}\right] = \tag{5.17}$$

$$\frac{\alpha}{\mathrm{dn}\,(e_0/\alpha,\,k)}\,[K(\sqrt{1-\delta^2}) + iK(\delta)]$$

$$2H = c_1 h\left(1-\frac{b}{a}\right) + c_1\left(1+\frac{b}{a}\right)\frac{\alpha K(\delta)}{\mathrm{dn}\,(e_0/\alpha,\,k)}$$

The equation of an equally strong outline L_0 in the fundamental period (for $-L < x < 0,\ y > 0$) can be written in the following parametric form on the basis of (5.11) and (5.12):

$$x + iy = -x_0 + \frac{1}{2}c_1\left(1-\frac{b}{a}\right)(\xi + e_0) + \frac{i\alpha c_1(1+b/a)}{2\mathrm{dn}\,(e_0/\alpha,\,k)}\int_1^t \frac{dt}{\sqrt{(t^2-\delta^2)(t^2-1)}}$$

$$\left(t = \frac{\mathrm{dn}\,(\xi/\alpha,\,k)}{\mathrm{dn}\,(e_0/\alpha,\,k)} = \sqrt{\frac{1-k^2\,\mathrm{sn}^2(\xi/\alpha,\,k)}{1-k^2\,\mathrm{sn}^2(e_0/\alpha,k)}}\right)$$

i.e., [16]

$$x = -x_0 + {}^1\!/_2 c_1(1-b/a)(\xi+e_0) \tag{5.18}$$

$$y = \frac{c_1\alpha(1+b/a)}{2\mathrm{dn}\,(e_0/\alpha,\,k)}F\left(\arcsin\sqrt{\frac{t^2-1}{t^2-\delta^2}},\ \delta\right)$$

$$(-e_0 \leqslant \xi \leqslant 0)$$

There remains to determine the constants σ_x^∞ and σ_y^∞. To do this, it is sufficient to use just the first two equilibrium equations (5.9) in this case, which can be written by using (2.15) thus:

$$\int_{E_3E_4}\sigma_y\,dx = (\sigma_x^\infty + \sigma_y^\infty)(L - x_0) + \int_{-1}^{-e_0}[F'(\zeta)+G'(\zeta)]\,d\zeta = 2LN_y \tag{5.19}$$

$$\int_{E_1E_4}\sigma_x\,dy = H(\sigma_x^\infty + \sigma_y^\infty) + i\int_{-1}^{-1+ih}[F'(\zeta)+G'(\zeta)]\,d\zeta = 2HN_x$$

Using the solution of the problem in the form (5.10) – (5.12), let us evaluate the integrals in (5.19)

$$\int_{-1}^{-e_0}[F'(\zeta)+G'(\zeta)]\,d\zeta = c_1(b-a)(1-e_0) + \frac{c_1\alpha(a+b)K(\sqrt{1-\delta^2})}{dn\,(e_0/\alpha,\,k)} \tag{5.20}$$

$$\int_{-1}^{-1+ih}[F'(\zeta)+G'(\zeta)]\,d\zeta = ic_1h(b-a) + c_1(a+b)\int_{-1}^{-1+ih}\frac{\mathrm{sn}\,(\zeta/\alpha,\,k)\,d\zeta}{\sqrt{\mathrm{sn}^2(\zeta/\alpha,\,k)-\mathrm{sn}^2(e_0/\alpha,\,k)}} =$$

$$ic_1h(b-a) + \frac{\alpha c_1(a+b)}{\mathrm{dn}\,(e_0/\alpha,\,k)}\int_\delta^0\frac{dt}{\sqrt{(t^2-\delta^2)(1-t^2)}} =$$

$$ic_1h(b-a) + \frac{i\alpha c_1(a+b)K(\delta)}{\mathrm{dn}\,(e_0/\alpha,\,k)}$$

Substituting the evaluated integrals into (5.19), we obtain the following relationships:

$$2(a+p)(L-x_0)+c_1(b-a)(1-e_0)+\frac{\alpha c_1(a+b)K(\sqrt{1-\delta^2})}{\operatorname{dn}(e_0/\alpha,k)}=2LN_y \quad (5.21)$$

$$2H(a+p)-c_1h(b-a)-\frac{\alpha c_1(a+b)K(\delta)}{\operatorname{dn}(e_0/\alpha,k)}=2HN_x$$

Using the second formula in (5.18), we find the hole diameter $2y_0$ in the $x=0$ section; we obtain

$$y_0=\frac{\alpha c_1(1+b/a)F(\varphi_0,\delta)}{2\operatorname{dn}(e_0/\alpha,k)} \quad (5.22)$$

$$\varphi_0=\arcsin\sqrt{\frac{t_0^2-1}{t_0^2-\delta^2}},\qquad t_0=\frac{1}{\operatorname{dn}(e_0'/\alpha,k)}$$

The four equations (5.16), (5.17) and (5.21) are to determine four constants. One of the constants remains a free parameter, as before.

The approximate solution of the problem considered was attempted by the small parameter method in [9, 11].

6. Application to the theory of a minimum weight structure. Let a structure or some element of it be a plate with holes in the plane state of stress. The plate thickness is considered constant. Let us assume that some ultimately admissible normal stress (taking account of a safety factor) determined from an elastic analysis of the structure is given. Let us note that plastic zones are not usually admitted in all structures desined for prolonged operation. For technological reasons the hole shape will usually be circular.

Under the assumptions mentioned, which are realized quite often in practice, the gain in weight of a structure obtained by using equally strong rather than standard circular holes can be easily estimated.

A comparison between equally strong and other holes shows that the stress σ_t therein is minimal compared with the maximum value of σ_t on any other hole outlines (*). In this sense, an equally strong hole possesses the property of greatest strength also (in comparison to all other holes).

The following hence results: a structure with equally strong holes possesses least weight in comparison to analogous structures without equally strong holes. Indeed, two analogous structures with differently shaped holes from plates of diverse thickness (naturally from the same material) will be equivalent in the strength sense if the maximal stress σ_t on the hole outline in these structures will be the same. Equally strong holes permit application of plates of least thickness for any given ultimately admissible stress.

We present specific estimates. Let an infinite plate of thickness h_0 with a load-free circular hole be subjected to homogeneous tension, which is described completely by the stress resulatants P and βP acting in the principal directions (β is some number, $\beta \leqslant 1$). Let us note the dimensionality of the stress resultant P, i.e. "the force divided by the length". In this case the maximal stress σ_t on the hole outline evidently is

$$\sigma_t^{\max}=(3-\beta)P/h_0$$

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*) This assertion should be considered as an intuitively obvious fact, verified in many cases. The author does not know of any rigorous proof.

According to (1.10), an analogous quantity in the case of an equal strength hole in a plate of thickness $h_{\min}$, is $\sigma_t^{\max} = (1+\beta)P / h_{\min}$

Equating these two quantities for structures of equivalent strength, we obtain

$$h_{\min} = (1+\beta) h_0 / (3-\beta)$$

Therefore, application of an equal-strength in place of a circular hole in this case permits a g% diminution in the weight of the considered element of the structure, where

$$g = 200\,(1-\beta) / (3-\beta) \tag{6.1}$$

without changing the strength. For example, if one principal stress resultant is twice the other, i.e. $\beta = 1/2$, use of the equally strong rather than the circular hole permits a 40% diminution in the plate weight while conserving the strength. It can be shown that g increases as the number of holes increases and boundaries are present. Hence, (6.1) can be considered as the lower bound of a possible diminution in the weight of a structure.

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STABILIZATION OF FREE ROTATION OF AN ASYMMETRIC TOP WITH CAVITIES COMPLETELY FILLED WITH A LIQUID

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E. P. SMIRNOVA
(Leningrad)
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We analyze the motion of an asymmetric top with cavities filled with a viscous incompressible liquid, and we study the stabilizing effect of the liquid on the rotation of the top around a given axis. The characteristic time for stabilization and the best orientation of the cavity relative to the solid body, have been found.

1. Equations of motion and their investigation. In a coordinate system whose axes are directed along the principal inertia axis of a body-liquid system, the equations of motion of a top for small Reynolds numbers reduce to the form [1]

$$I\omega^{\cdot} + [\omega, I\omega] = \frac{\rho}{\nu}\{P\omega^{\cdot\cdot} + [\omega, P\omega^{\cdot}]\} \tag{1.1}$$

Here I is the system's inertia tensor, ω is the top's angular velocity, ρ and ν are the liquid's density and kinematic viscosity, respectively. The right-hand side of (1.1) describes the force moment caused by the liquid's motion relative to the top; terms of a higher order of smallness with respect to ρ / ν are discarded. The tensor $P = \| P_{ij} \|$ is determined only by the shape of the cavity and is symmetric, $P_{ii} > 0$. In the case of several cavities this tensor equals the sum of the tensors for the individual cavities. The computation of the components of this tensor for a given cavity is a separate problem. It has been obtained in [1] for cavities of certain shapes. The motion of a solid with a symmetric cavity, when the tensor P is a multiple of the unit tensor, was studied in [1]. Here we examine the case of an arbitrary tensor P.

We rewrite Eq. (1.1) in the form

$$\mathbf{M}^{\cdot} + [\omega, \mathbf{M}] = 0, \qquad \mathbf{M} = I\omega - \frac{\rho}{\nu} P\omega^{\cdot}$$

where $\mathbf{M}$ is the system's total impulse moment. Hence right away we see the two relations

$$\mathbf{M}\mathbf{M}^{\cdot} = 0, \qquad \mathbf{M}^{\cdot}\omega = 0 \tag{1.2}$$

i.e. the law of preservation of the impulse moment and the law of dissipation of the system's energy

$$\Gamma^2 - 2\frac{\rho}{\nu}(\Gamma, P\omega^{\cdot}) = \text{const} \tag{1.3}$$

$$\frac{dE}{dt} = \frac{dH}{dt} - \frac{\rho}{\nu}\frac{d}{dt}(\omega, P\omega^{\cdot}) = -\frac{\rho}{\nu}(\omega^{\cdot}, P\omega^{\cdot}) < 0$$

$$E = \frac{1}{2}(\omega, I\omega) - \frac{\rho}{\nu}(\omega, P\omega^{\cdot})$$